Examiner's commentary

Writing a 4,000-word essay is a huge task for a student. The burden of this is made much easier if a student finds a topic that they genuinely feel personally interested in and passionate about. In this essay, we are not told if the student is from Bermuda (the student should have said), but the student does say so in the Reflection Document. This makes the personal motivation clear. So, a good choice of topic for the student, and the Mathematics involved is well within the grasp of a good HL Maths student. At times, mathematical ideas are not fully explained with over reliance on references. It is important that an examiner is confident that the student fully understands the mathematics used. For this reason, essays that use mathematics too far beyond the syllabus often don't score so well. The reflections show a high degree of personal engagement – naturally, as this was a topic of such interest to the candidate.

Mathematics Extended Essay

Word Count: 3947 words

Numerical Solution of a Logistic Growth Model for the Population of Lionfish in the Atlantic Ocean

How informative and accurate is implementing the numerical method onto the logistic equation of the growth of the lionfish population in the Atlantic Ocean?

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Introduction

Research Question: How informative and accurate is implementing the numerical method onto the logistic equation of the growth of the lionfish population in the Atlantic Ocean?

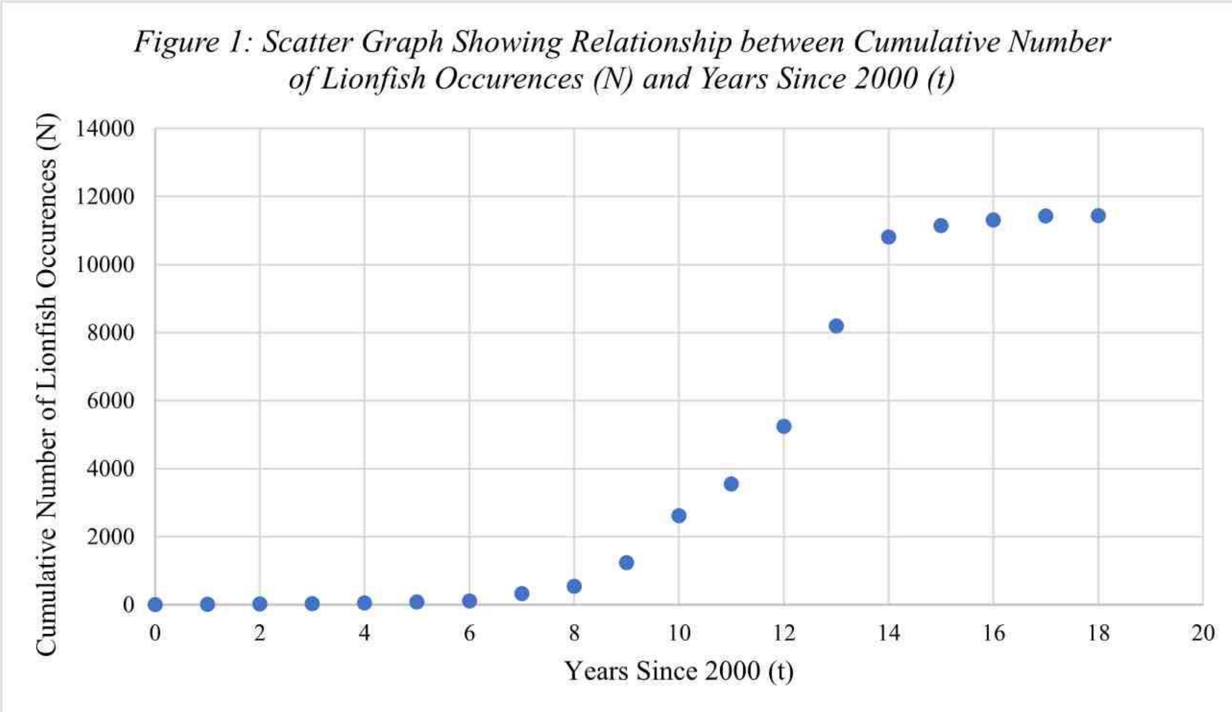
Background: After their first sighting in Bermuda (1999), (Bermuda Lionfish Task force) lionfish have become an invasive species in the Atlantic Ocean. According to The Bermuda Lionfish Task Force, the problem lies in their rapid reproduction rate, "reproducing every 3 to 4 days year-round" which "could result in 2,000,000 eggs laid in one year" (Bermuda Lionfish Task force). The "infestation" (Linendoll) is almost unstoppable in Bermuda's marine ecosystem because of their resistance to parasites, ability to consume 30 times their stomach volume and safety from any predators (Bermuda Lionfish Task force). From the international news source Cable News Network to the local Bermudian Ocean Support Foundation, the population growth of lionfish has become a crisis in need of intervention.

Hypothesis: To model the rapid population growth of lionfish in the Western Atlantic Ocean, a logistic differential equation will be created. "A differential equation is a function that represents the rate of change of one variable in relation to another variable" (Neumann). Therefore, a differential equation can be used to model the growth of lionfish population since it expresses the rate of growth with respect to time. I predict that the initial population size and model parameters from sample data should provide a numerical solution that will be accurate and relatively informative on the population growth of lionfish in the Atlantic Ocean. I will produce my own function modelling the growth of the population of lionfish and consequently make comparisons using error analysis to evaluate the accuracy of my function against a computer-generated function. The accuracy of my function will be dependent upon percentage errors calculated from the comparison. The creation of my own function will involve using Euler's method, a method that uses initial observed data to make future predictions.

Data

Table 1: The Cumulative Number of Reported Occurrences of Lionfish from 2000 to 2018 in the Atlantic Ocean (Non Indigenous Aquatic Species - USGS)

$Time\ (t)\ in\ years\ since\ 2000$	$Cumulative\ Number\ of\ Lionfish\ Occurrences\ (N)$
0	4
1	11
2	16
3	24
4	48
5	80
6	104
7	323
8	540
9	1232
10	2614
11	3548
12	5241
13	8196
14	10806
15	11141
16	11304
17	11430
18	11434



Discussion

Initially, the population was assumed to follow an exponential growth curve, which can be observed on Figure 1 between 2000 to 2013 as it is concave up until the point of inflection on the graph, which can be observed at approximately $t = 13$ years. The true point of inflection will be calculated further in the paper. Exponential growth is defined as when "growth rate stays the same regardless of population size, making the population grow faster and faster as it gets larger" (Khan Academy).

Mathematician P. H. Verhulst explored population growth and suggested (Springer Link) an "increase of the population is… limited by the size and the fertility of the country". In other words, external factors are unavoidable in the real world and impact sustainability of a growth pattern over time. Hence, during population growth, the population will become more stable as a result of these inevitable constraining factors. In terms of lionfish, some external influences are indirect or direct. For example, creating lionfish taskforces to eliminate the lionfish population is more indirect as it provides awareness to the population. Meanwhile, more directly, schools can encourage Bermudian students to participate in competitions to fish the lionfish more aggressively. Collectively, these external factors will cause the shape of the exponential graph to change. Consequently, the graph does not fit the model of exponential growth, but rather a logistic growth.

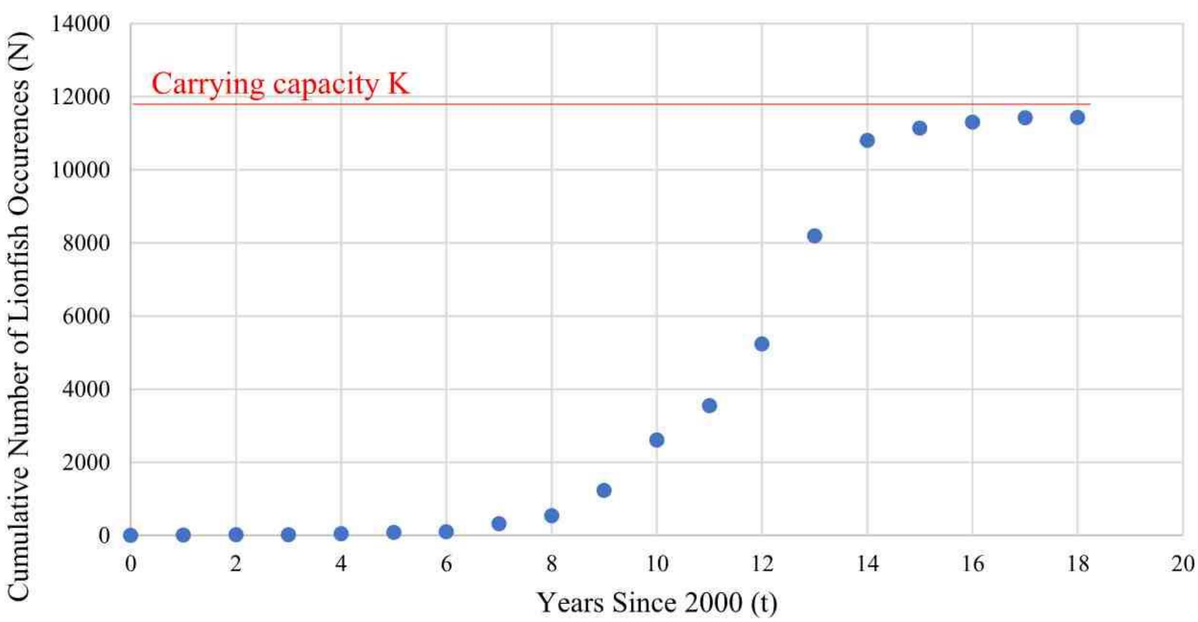
Producing Differential Equation for Logistic Function

By inspection of the graph, the data points increase slowly as time increases between 0 and 8 years, then the points increase at a quick pace between 8 to 14 years and plateau through the end, after around 16 years. Through observation of the graph, it appears probable that the population growth of lionfish follows a sigmoidal shape. In order to confirm this assumption, the relationship between the population growth of lionfish will be represented as a differential equation. The two variables are the cumulative number of lionfish occurrences (N) and years since 2000 (t).

If we look at N(t) as a logistic function, we can develop a differential equation. To begin, the growth rate can be expressed as $\frac{dN}{dt}$ where dN means the change in population of lionfish and dt means the change in time. As stated in the discussion above, the basis of the logistic curve begins with deriving an exponential function. The general exponential differential equation is $\frac{dN}{dt} = rN$ where r is a proportionality constant and N is the population of lionfish. However, the equation must be modified to transform the exponential function to a logistic function (Khan

Academy). Exponential growth is where "growth rate does not change even if the population gets very large" whereas logistic growth is when the "growth rate gets smaller as the population approaches its maximum size" (Khan Academy). From the S-shaped curve below, the carrying capacity (World Population History) can be identified as the plateau (a sort of asymptote) – this carrying capacity shows that the population of lionfish genuinely cannot increase forever as a result of external factors.

Figure 2: Scatter Graph Showing Relationship between Cumulative Number of Lionfish Occurences and Time



It should also be assumed that when $\frac{dN}{dt} = 0$, the gradient of the graph is 0, which can be graphically represented as a horizontal line. This is the definition of the carrying capacity, where we reach "the maximum number of a species an environment can support indefinitely" (World Population History) which is denoted by the letter K.

Therefore, we can multiply rN with $\left(\frac{K-N}{K}\right)$ which represents the "fraction of the carrying capacity that has not yet been 'used up'" (Khan Academy). This can then be simplified to

$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$. As N approaches K, then $\left(1 - \frac{N}{K}\right)$ will approach 0 since $\left(1 - \frac{K}{K}\right) = 0$. As N reduces to a value smaller than K, then $\left(1 - \frac{0}{K}\right)$ will approach 1. Thus, the differential equation for a logistic function can be written as $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$ where r is a proportionality constant and K is the carrying capacity.

Estimating Parameters r and K from Sample Data

In order to draw slope fields and implement Euler's method for the differential equation, the parameters □ and □ from the logistic differential equation must be estimated (Khan Academy). The equation must be rearranged as shown below:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

$$\frac{\left(\frac{dN}{dt}\right)}{N} = r\left(1 - \frac{N}{K}\right)$$

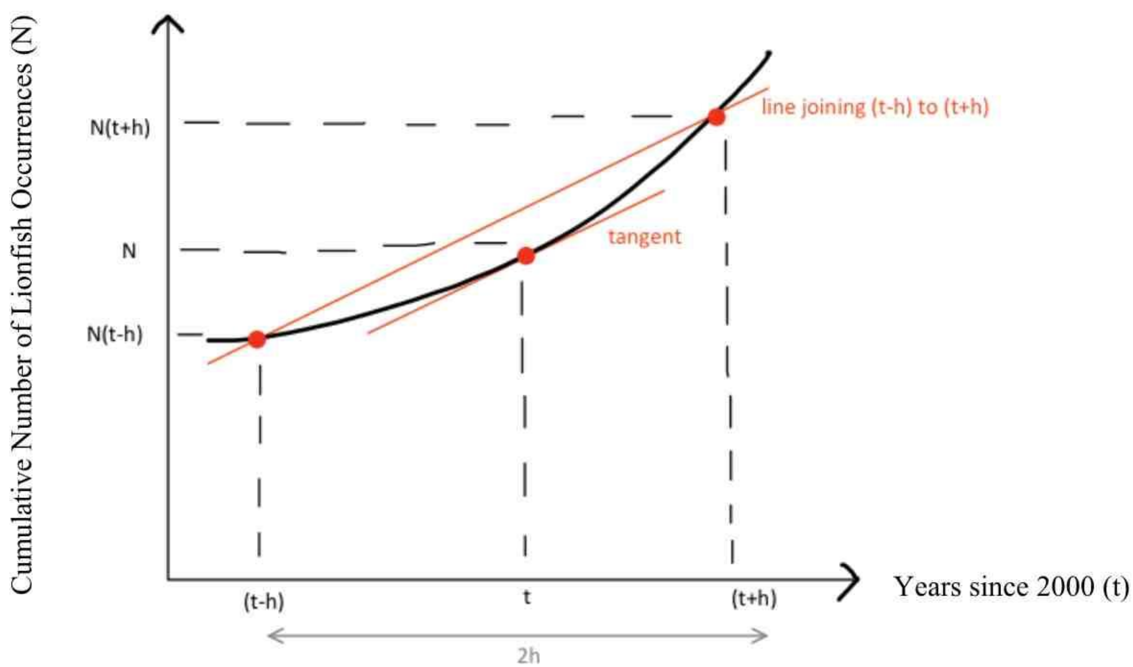
$$\frac{\left(\frac{dN}{dt}\right)}{N} = r - \frac{rN}{K}$$

$$\frac{dN}{dt}\left(\frac{1}{N}\right) = r - \frac{rN}{K}$$

$$\frac{dN}{dt}\left(\frac{1}{N}\right) = -\frac{rN}{K} + r$$

$\frac{\frac{dN}{dt}}{N}$ represents the relative growth rate and it is a linear function of N. Essentially, this represents the rate at which the lionfish population is growing (the number of fish reproduced by each lion fish during each unit of time) relative to the population (N). Since it is linear, the gradient and y intercept of the straight line can be calculated. The gradient is $-\frac{r}{K}$ and the y intercept is r. Also known as the symmetric difference quotient, this approach is used to approximate the gradient of a specific point using the two surrounding points equidistant from that specific point.

Figure 3: A Diagram Illustrating the Symmetric Difference Calculation



In order to estimate the growth rate of lionfish population represented by $\frac{dN}{dt}$, the following formula is used: $\frac{dN}{dt} = \frac{N(t+h)-N(t-h)}{2h}$

The approach for the estimation of the relative growth rate can be done in two steps.

Step 1: Calculate the growth rate at, for example, time $t = 12$ years

$$\frac{dN}{dt} = \frac{N(12+1) - N(12-1)}{2\times 1} = \frac{8196 - 3548}{2\times 1} = 2324$$

Step 2: Calculate the relative growth rate

$$\frac{dN}{dt}\left(\frac{1}{N}\right) = (2324)\left(\frac{1}{5241}\right) = 0.4434268269$$

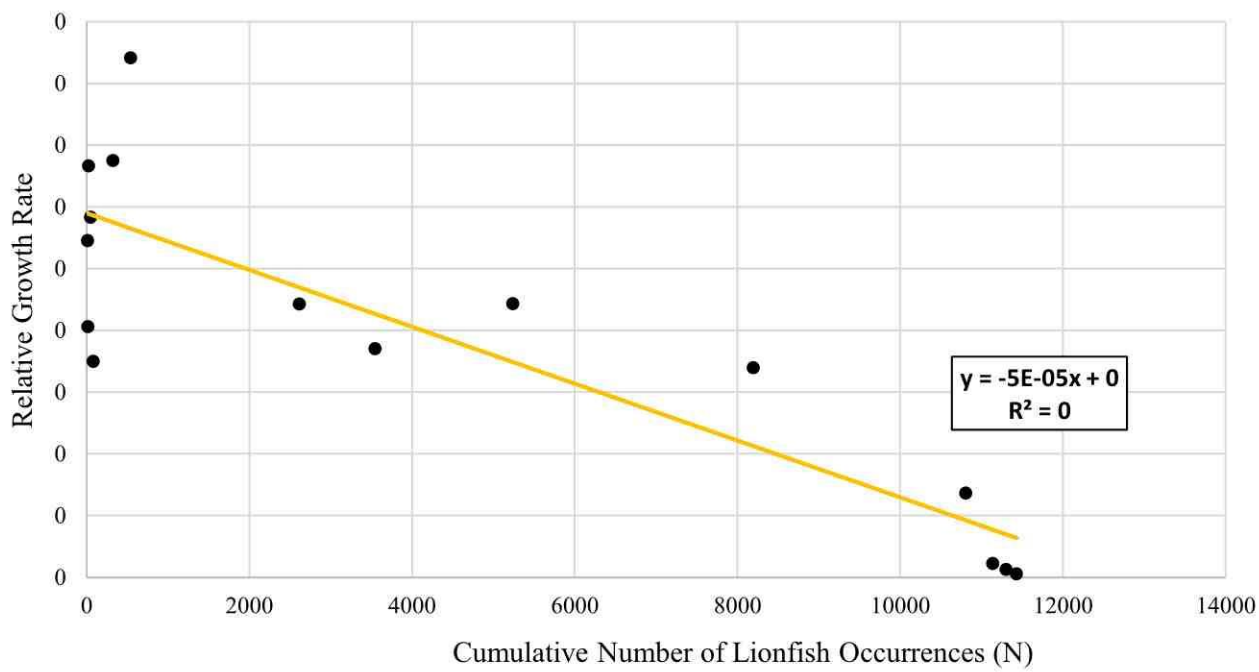
Following is a table displaying the estimated values for the number of fish reproduced by each lionfish for each unit of time:

Table 2: Cumulative Number of Lionfish and Relative Growth Rate of Lionfish

Time (t) *in years since* 2000	*Cumulative Number of Lionfish* (N)	*Relative Growth Rate of Lionfish* $(\frac{dN}{dt})\left(\frac{1}{N}\right) = \frac{\left(\frac{(N(t+1)-N(t-1))}{2}\right)}{N}$
0	4	[redacted]
1	11	0.545454545
2	16	0.406250000
3	24	0.666666667
4	48	0.583333333
5	80	0.350000000
6	104	1.168269231
7	323	0.674922601
8	540	0.841666667
9	1232	0.841720779
10	2614	0.442999235
11	3548	0.370208568
12	5241	0.443426827
13	8196	0.339494876
14	10806	0.136266889
15	11141	0.022349879
16	11304	0.012783086
17	11430	0.005686789
18	11434	[redacted]

By observation, it can be seen that at times of $t = 0, 6, 7, 18$, the estimated value is either an outlier or does not produce an answer. Hence, to achieve more accurate results, the highlighted data points will be omitted from the corresponding graph below.

Figure 4: A Graph of Cumulative Number of Lionfish Occurences (N) against Relative Growth Rate of Lionfish



The graph, after omitting $t = 0, 6, 7, 18$ (which produced inconclusive results), displays a linear equation of $y = -0.00005x + 0.5901$ with an R^2 coefficient of 0.7571. The value of parameter r can be deduced from the linear equation and the value of the R^2 coefficient implies that the linear graph has a relatively good fit to the data (The Analysis Factor). In order to determine the parameter r, the y intercept must be found. At $x = 0,\ r = 0.5901$. To find parameter K, the carrying capacity, both the gradient of the linear equation and r are required.

$$-\frac{r}{K} = -0.00005$$

$$-\frac{0.5901}{K} = -0.00005$$

$$K = 11802$$

Therefore, the parameters are:

$K = 11802$ and $r = 0.5901$

Hence, the slope field can be found for the differential equation below:

$$\frac{dN}{dt} = 0.5901N\left(1 - \frac{N}{11802}\right)$$

Slope Field for Differential Equation

Slope fields are an excellent way to graph a family of functions for differential equations without knowing the solutions explicitly. The steps for drawing slope fields by hand are shown below:

Step 1: Set the derivative in the differential equation to a constant, for example $\frac{dN}{dt} = m$ where m represents the gradient at certain points on the tN-plane.

Step 2: Draw this family of equations, known as isoclines, for different values of m. Isoclines are produced when $\frac{dN}{dt} = m$ as the gradient. The plot of 'hatch marks' forms the slope field (Math Surgery).

In the same way, a family of functions for the differential equation $\frac{dN}{dt} = 0.5901N\left(1 - \frac{N}{11802}\right)$ can be illustrated using slope fields. However, I will use Bluffton University's graphing application to produce slope fields from the derived differential equation. Since both time and population cannot have negative values, the domain was set to $0 \leq t \leq 20$ and the range set to $0 \leq N \leq 14000$.

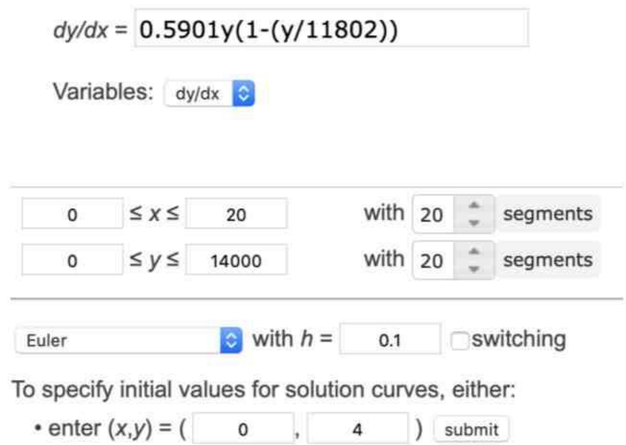


Figure 5: Bluffton University Slope Field Parameters

The corresponding slope field produced is shown (Bluffton University):

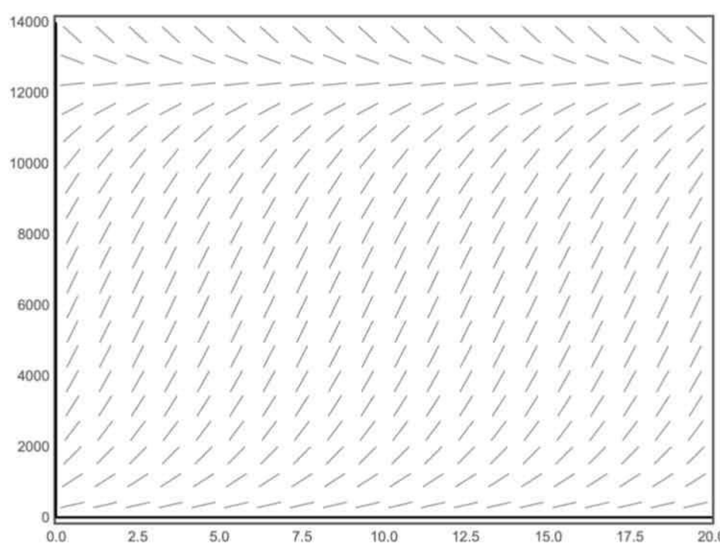


Figure 6: Bluffton University Slope Field Pattern

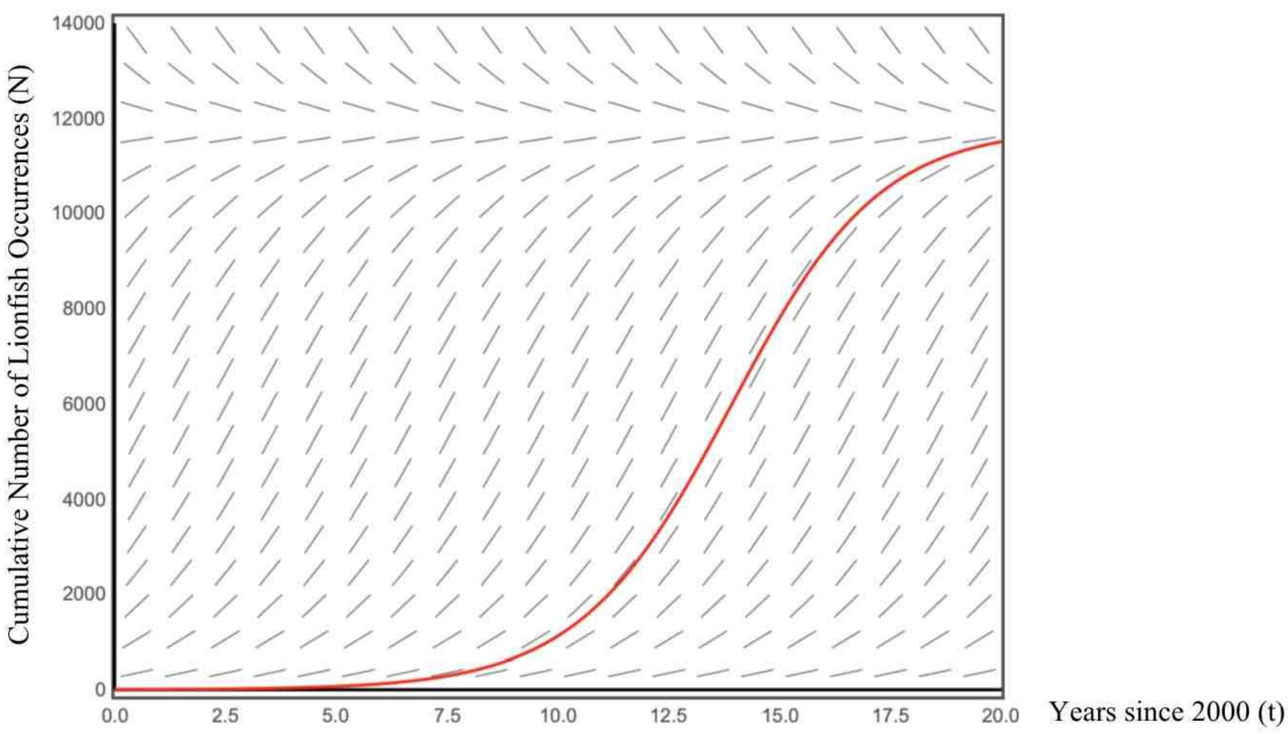
The slope field gives an idea of what solutions to the logistic function look like. In fact, it provides a family of functions that are solutions to the differential equation $\frac{dN}{dt} = 0.5901N\left(1 - \frac{N}{11802}\right)$. According to the slope field, the family of functions appear to converge at the $K = 11802$, the carrying capacity. Alternatively, the slopes point generally upwards when the population is less than the carrying capacity and the slopes point generally downwards when the population is greater than the carrying capacity.

From the slope field, we can then graphically construct an approximate solution curve that corresponds to a given initial condition. To do so we have to make sure the solution curve follows two rules:

a) It follows the direction of the hatch marks at every point
b) At no point will the different solutions to the differential equation intersect (never cross)

The initial point based on the sample data is (0, 4) hence the corresponding particular solution $\frac{dN}{dt} = 0.5901N\left(1 - \frac{N}{11802}\right)$ intersecting (0,4) is illustrated below.

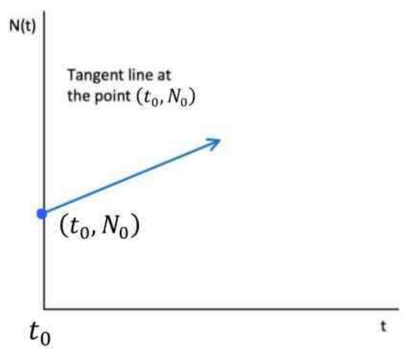
Figure 7: A Graph Illustrating the Slope Field produced from the Approximated Solution of a Derived Logistic Function Equation (Bluffton University)



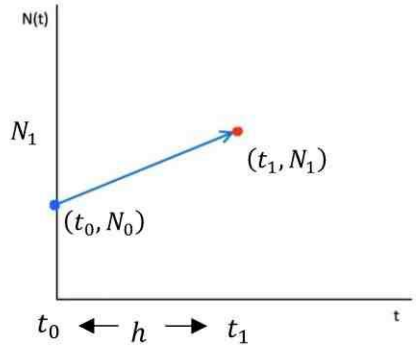
Employing Euler's Method (Numerical Solution) for Differential Equation

"Euler's method is a numerical tool for approximating values for solutions of differential equations" (Khan Academy). It is employed when analyzing a differential equation, which uses the idea of linear approximation, where tangent lines are drawn to approximate the solution to an initial-value problem (Calc Workshop). The basis of Euler's Method begins with a known point (x_0, y_0), a step interval h and a differential equation.

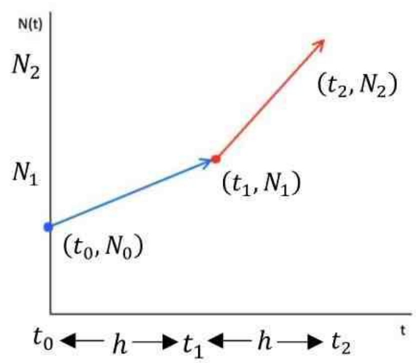
My goal for this paper is to approximate values for a particular solution of the differential equation $\frac{dN}{dt} = 0.5901N\left(1 - \frac{N}{11802}\right)$ which can also be written as $f(t, N) = 0.5901N\left(1 - \frac{N}{11802}\right)$. Since the gradient $\frac{dN}{dt}$ indicates the direction in which the solution curve points, I can reconstruct the graph of the solution using tangent lines, as follows.



Step 1: Begin with a starting point (t_0, N_0). The equation of a tangent through this point can be written as $N - N_0 = m(t - t_0)$ where m is the gradient at (t_0, N_0) which can be written as $f(t_0, N_0)$. Thus, the equation of a tangent through a point (t_0, N_0) is given as $N - N_0 = f(t_0, N_0)(t - t_0)$ or $N = f(t_0, N_0)(t - t_0) + N_0$.



Step 2: By setting $t = t_1$ and $N = N_1$ the equation above turns into $N_1 = hf(t_0, N_0)+N_0$ where h is the step size, represented as $h = t_1 - t_0$. This yields the Euler approximation at N_1.



Step 3: The equation of a tangent through a point (t_1, N_1) is given as $N_2 = f(t_1, N_1)(t_2 - t_1)+N_1$ which can also be written as $N_2 = hf(t_1, N_1)+N_1$ because the step size is $h = t_2 - t_1$. This yields the Euler Approximation at N_2.

Euler's method is a recursive process and by continuing, the sequence of approximations to the differential equation $\frac{dN}{dt} = f(t, N)$ at the points $t_{n+1} = t_n + h$ can be found using the formula, $N_{n+1} = hf(t_n, N_n)+N_n$. Now consider the differential equation $\frac{dN}{dt} = f(t, N)$ where $f(t, N) = 0.5901N\left(1 - \frac{N}{11802}\right)$ which was derived previously, for the population of lionfish, it is known that at 0 years (in 2000), there were 4 lionfish occurrences hence $(t_0, N_0) = (0,4)$. A way to display calculations for approximate values for a particular solution of the differential equation is using a table of values. It is important to note that Euler's method will have inaccuracies as the step size must be large enough to produce an accurate line, but also not too small where the approximation is under or overestimated. In this situation, I will use a step interval of 0.5.

Table 3: Calculations involving Euler's Method

	t_n	N_n	$f(t_n, N_n) = m_n$
$n = 0$	$t_0 = 0$	$N_0 = 4$	$= 0.5901(4)\left(1 - \frac{4}{11802}\right) = 2.3596$
$n = 1$	$t_1 = 0.5$	$N_1 = 4 + (0.5 \times 2.3596)$ $= 5.1798$	$= 0.5901(5.1798)\left(1 - \frac{5.1798}{11802}\right)$ ≈ 3.05526
$n = 2$	$t_2 = 1$	$N_2 \approx 5.1798 + (0.5 \times 3.05526)$ ≈ 6.7074	$= 0.5901(6.7074)\left(1 - \frac{6.7074}{11802}\right)$ ≈ 3.95579
$n = 3$	$t_3 = 1.5$	$N_3 \approx 6.7074 + (0.5 \times 3.95579)$ ≈ 8.6853	$= 0.5901(8.6853)\left(1 - \frac{8.6853}{11802}\right)$ ≈ 5.12142
$n = 4$	$t_4 = 2$	$N_4 \approx 8.6853 + (0.5 \times 5.12142)$ ≈ 11.2460	$= 0.5901(11.2460)\left(1 - \frac{11.2460}{11802}\right)$ ≈ 6.6299

From the table, we can see the approximate cumulative number of lionfish after two years is $11.2460 \approx 11$. The actual value from sample data is 16, so this is not a bad estimate. However, the differential equation $f(t, N) = 0.5901N\left(1 - \frac{N}{11802}\right)$ can easily be solved for an exact solution so I will solve it to check the accuracy of Euler's method. In problems like this one, it is important to know the accuracy of one's approximation and if the method is reliable in providing a good approximation by checking the formula estimation with actual observed data.

Solving the General Logistic Equation

In order to solve a differential equation, it must be integrated. Integration involves doing antidifferentiation, where the derivative is transformed into the original function. Since N is a function of t, the equation must be split to have all N values on one side of the equation. It is important to note that in this case, implicit differentiation will be used since two different variables are present. Additionally, the variables must be separated. We know that $N(0) = 4$ (patrickJMT).

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

Expand the equation:

$$\frac{dN}{dt} = \frac{rN(K - N)}{K}$$

Multiply by dt on both sides, then divide both sides by $N(K - N)$ to write in differential form:

$$\frac{dN}{N(K - N)} = \frac{r}{K}dt$$

Rearrange in order to have only r on one side of the equation:

$$\frac{dN\ (K)}{N(K - N)} = rdt$$

Integrate each side:

$$\int \frac{K}{N(K - N)}(dN) = \int (r)dt$$

The algebraic fraction in the integral is easier to evaluate when it is separated or split, which can be done by using partial fractions. A and B will have different values:

$$\frac{K}{N(K - N)} = \frac{A}{N} + \frac{B}{K - P}$$

To find the values for A and B, multiply $\frac{K}{N(K-N)}$ by $N(K-N)$ on both sides to eliminate the denominators attached to A and B. Then simplify:

$$(N(K-N))\frac{K}{N(K-N)} = \left(\frac{A}{N} + \frac{B}{K-N}\right)(N(K-N))$$

If we let $N = 0$,

$$K = A(K-N) + BN$$

$$K = A(K-0) + B0$$

$$K = AK$$

$$A = 1$$

If we let $N = K$,

$$K = A(K-N) + BN$$

$$K = A(K-K) + BK$$

$$K = BK$$

$$B = 1$$

Hence, $\int \frac{1}{N} + \frac{1}{K-N}(dN) = \int (r)dt$

And the right-hand side will be solved directly (where c is a constant as there exists a family of functions):

$$tr + c$$

By using partial fractions, the left-hand side of the integral can be more easily broken down into natural logarithms which can then be simplified into one logarithm:

$$\ln|N| + \ln|K - N| = tr + c$$

Due to chain rule, $\ln|K - N|$ will become negative (Smith):

$$\ln|N| - \ln|K - N| = tr + c$$

The entire equation can be rewritten as:

$$\ln\left|\frac{K - N}{N}\right| = -tr - c$$

Eliminating the logarithms will involve exponentiation of both sides:

$$e^{\ln\left|\frac{K-N}{N}\right|} = e^{-tr-c}$$

$$\left|\frac{K - N}{N}\right| = e^{-tr}e^{-c}$$

Allowing $e^{-c} = A$, $\frac{K-N}{N} = e^{-tr}A$

Rearrange the equation to solve for N:

$$\frac{K}{N} - \frac{N}{N} = (e^{-tr}A)$$

$$\frac{K}{N} - 1 = (e^{-tr}A)$$

$$\frac{K}{N} = (e^{-tr}A) + 1$$

$$N = \frac{K}{1 + Ae^{-rt}}$$

If we let $t = 0$, then $N = N(0)$, the initial population. Using this new value of t, replace into the equation for $\frac{K-N}{N} = Ae^{-tr}$:

$$\frac{K - N_0}{N_0} = Ae^{-(0)r}$$

$$\frac{K - N_0}{N_0} = A$$

As a result, the solution to the logistic equation is:

$$N(t) = \frac{K}{1 + Ae^{-rt}}, where \ \frac{K - N_0}{N_0} = A$$

Solving Specific Logistic Equation to Find Particular (Analytical) Solution

Once the general logistic equation is found, a specific solution can be derived using the sample data. According to the equation, $N(t) = \frac{K}{1+Ae^{-rt}}$, parameters K, r and A are required to find the specific solution. Parameter K, the carrying capacity, was found originally earlier: $K = 11802$. We also know, according to initial data, that $N_0 = 4$ so for A:

$$A = \frac{K - N_0}{N_0}$$

$$A = \frac{11802 - 4}{4} = 2949.5$$

Since, $N(t) = \frac{K}{1+Ae^{-rt}}$

Hence, $N(t) = \frac{11802}{1+2949.5e^{-rt}}$

We can now input a point from the original data to solve for parameter r. For example, at time $t = 13$ years, $N = 8196$ the cumulative number of lionfish occurrences.

$$N(13) = 8196 = \frac{11802}{1 + 2949.5e^{-r(13)}}$$

$$1 + 2949.5e^{-r(13)} = \frac{11802}{8196}$$

$$2949.5e^{-r(13)} = \frac{11802}{8196} - 1$$

$$2949.5e^{-r(13)} = 1.439970717 - 1$$

$$e^{-r(13)} = \frac{0.4399707174}{2949.5}$$

$$\ln\left(\frac{0.4399707174}{2949.5}\right) = -13r$$

$$\frac{\left(\ln\left(\frac{0.4399707174}{2949.5}\right)\right)}{-13} = r = 0.6777260038$$

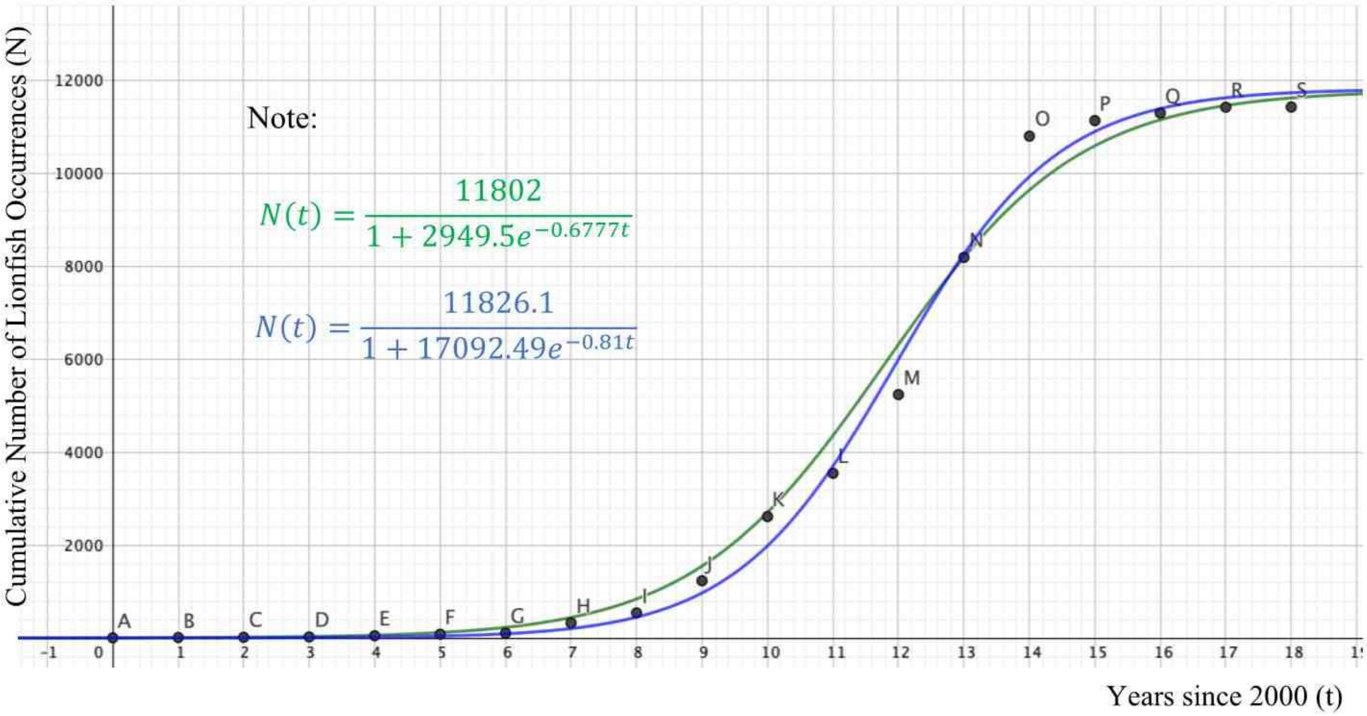
$$r \approx 0.6777$$

If the r is replaced into the specific solution of the logistic function, a particular function arises which can be plotted against the original data. The function is:

$$N(t) = \frac{11802}{1 + 2949.5e^{-0.6777t}}$$

For purposes of comparison, I used GeoGebra, a graphing application, to fit a logistic curve onto the sample data. The regression curve is shown:

Figure 8: Graph of Technological Fit of Sample Data (Blue) Compared to Analytical Solution (Green)



It seems that the logistic regression curve is a better approximation of sample data. However, I will use my function (in green) because it accurately passes through the original point, as seen in Figure 9.

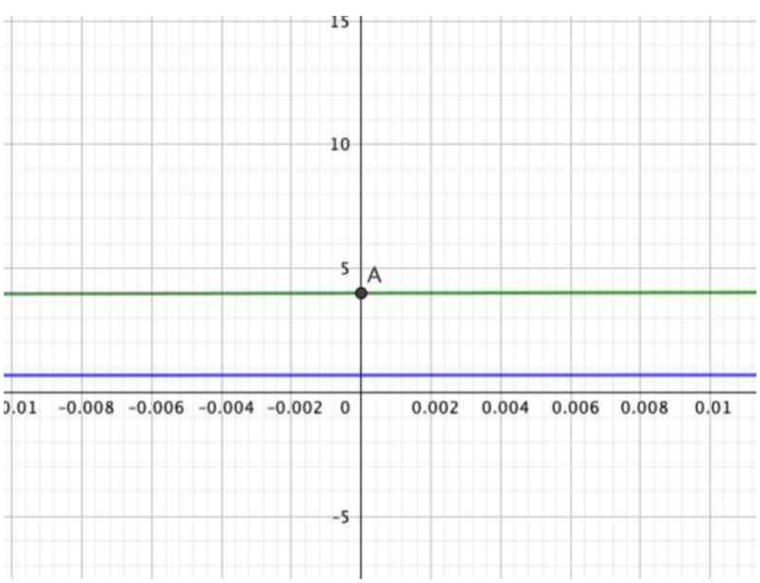


Figure 9: Logistic regression curve intercept compared to my function's intercept

Note:

$$N(t) = \frac{11802}{1 + 2949.5e^{-0.6777t}}$$

$$N(t) = \frac{11826.1}{1 + 17092.49e^{-0.81t}}$$

Comparison between Particular (Analytical) Solution and Euler's (Numerical) Approximation

I will use a spreadsheet to compare both solutions because it is easier to perform Euler's method computations using technology. I will show Euler's approximations as well as the analytical solution at the given times, for different step sizes for $h = 0.5$, $h = 0.1$, $h = 0.05$ and $h = 0.005$.

Errors in Euler's Method (Numerical Solution)

The analytical solution was determined by inputting the time (t) into the function I previously determined ($N(t) = \frac{11802}{1+2949.5e^{-0.6777t}}$). Through substitution of the time, the analytical solution is found. Additionally, the numerical solutions produced with the different step sizes are found through the use of a spreadsheet. For example, if a time of $t_1 = 1$ is selected, the resulting answer is $N(t) = \frac{11802}{1+2949.5e^{-0.6777(1)}} = 7.874785$. The same method is used for all times in the analytical solution function. If we examine a step size of $h = 0.5$, a time of $t_1 = 1$ is selected. Hence, $N(t) = \frac{11802}{1+2949.5e^{-0.6777(0.5)}} = 6.707429$. The same method is used for all step sizes.

Table 4: Exact Solution of Function with Different Step Sizes

Years after 2000, t_n	Analytical Solution (N)	$h = 0.5$	$h = 0.1$	$h = 0.05$	$h = 0.005$
0	4.000000	4.000000	4.000000	4.000000	4.000000
1	7.874785	6.707429	7.095010	7.153464	7.208453
2	15.498129	11.246053	12.582380	12.790396	12.987623
3	30.482331	18.851973	22.306170	22.860866	23.390916
4	59.880247	31.591274	39.520814	40.833606	42.097743
5	117.347610	52.909341	69.946237	72.851243	75.669477
6	228.891594	88.529369	123.562457	129.704911	135.705650

From Table 4, it can be observed that different step sizes produce different results. A smaller step size produces a more accurate projection as it is closer to the analytical solution. This can be shown through the comparison of two step sizes. For example, the value (N) associated with $h = 0.5$ at $t_2 = 2$ is $N = 11.246053$ can be compared to the same time with a different step size of $h = 0.05$, whereby the spreadsheet shows $N = 12.790396$. These values can be compared to the analytical solution of $N = 15.498129$. The $h = 0.05$ value is closer to the analytical solution. I will create a table showing the percentage errors associated with the analytic solution and the different numerical solutions with different step size. Since Euler's method uses tangent lines to approximate the function from the initial value, the tangent lines are only a fair approximation over a small interval. This error in approximation can be bigger if the exact function is concave up which leads to an underestimation of Euler's method for example from $t = 0$ to $t = 6$ years after. The error also increases as the distance from the original point increases. Through comparison of step sizes, it can be seen that a smaller step size produces a more accurate and precise approximation.

Table 5: Percentage Errors Associated with Different Step Sizes

t_n	$h = 0.5$	$h = 0.1$	$h = 0.05$	$h = 0.005$
$t_0 = 0$	0.000%	0.000%	0.000%	0.000%
$t_1 = 1$	7.711%	1.038%	0.479%	0.044%
$t_2 = 2$	14.824%	2.064%	0.956%	0.088%
$t_3 = 3$	21.385%	3.081%	1.431%	0.133%
$t_4 = 4$	27.436%	4.086%	1.904%	0.177%
$t_5 = 5$	33.014%	5.081%	2.374%	0.221%
$t_6 = 6$	38.154%	6.066%	2.842%	0.265%

There is a direct relationship between the size of the step and the size of the percentage error. For example, the percentage error with a step size of $h = 0.05$ at $t_3 = 3$ is 1.431%. If the step size decreases by a factor of ten, thus $h = 0.005$, then the percentage error decreases by a factor of ten. The resulting percentage error is 0.133%. However, regardless of the step size, as time increases, the percentage error increases. This is due to the accumulation of errors, since the approximation becomes less and less accurate.

The Point of Inflection

The fastest rate of growth can be found by finding the point of inflection on the curve, i.e. the middle of the S curve. In other words, the point of inflection occurs where the second derivative equates to zero. At the point of inflection, $\frac{d^2N}{dt^2} = 0$.

$$\frac{dN}{dt} = 0.5901N\left(1 - \frac{N}{11802}\right)$$

$$\frac{dN}{dt} = 0.5901N - \frac{0.5901N^2}{11802}$$

$$\frac{dN}{dt} = 0.5901N - 0.00005N^2$$

$$\frac{d^2N}{dt^2} = 0.5901 - 0.0001N$$

$$\frac{d^2N}{dt^2} = 0 = 0.5901 - 0.0001N$$

$$0.0001N = 0.5901$$

$$N = \frac{0.5901}{0.00005} = 5901$$

The point of inflection occurs at 5901 lionfish which coincides with a time that can be approximated once 5901 is substituted into the following equation:

$$N(t) = 5901 = \frac{11802}{1 + 2949.5e^{-rt}}$$

$$1 + 2949.5e^{-rt} = \frac{11802}{5901}$$

$$2949.5e^{-rt} = 2 - 1$$

$$2949.5e^{-0.6777t} = 1$$

$$e^{-0.6777t} = 3.390405153 \times 10^{-4}$$

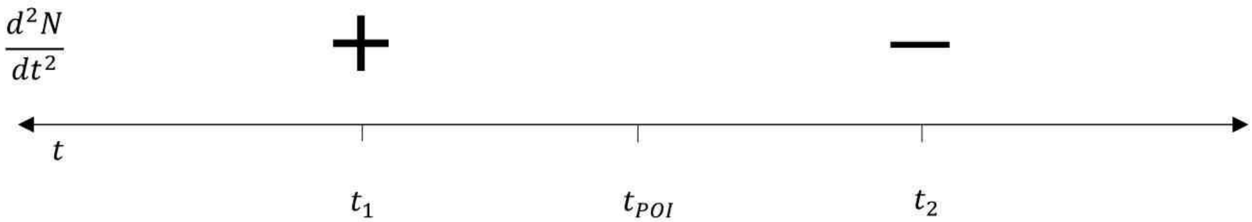
$$\ln(3.390405153 \times 10^{-4}) = -0.6777t$$

$$-7.989390944 = -0.6777t$$

$$\mathrm{t} = 11.78897882$$

$$\mathrm{t} \approx 11.79$$

Hence, the point of inflection occurs at a time of 11.79 years. In order to confirm this is the point of inflection, a sign diagram can be used. Additionally, the concavity of the graph can be determined using the sign diagram.



A positive sign denotes concave up where the slope is increasing. A negative sign denotes concave down where the slope is decreasing. Concave down indicates that the growth rate is reducing. Something must have caused the growth to slow down, perhaps the creation of lionfish groups. According to Dive Training, "the best available option is removing lionfish from reefs" (Dive Training). Many associations are hosting spearing competitions to encourage lionfish fishing to make the public aware of the issue. Currently, a local group called Guardians of the Reef hosted a Lionfish Derby. Fortunately, many teams competed and "a total of 103 lionfish" were captured in 4 days. However, lionfish are easiest to catch at certain times of the year, meaning these derbies cannot take place year-round. Additionally, this method may be inefficient in light of the grand scheme of lionfish (due to its vast size, 103 does not appear to make a difference). According to Dr. Corey Eddy, the Guardians of the Reef Scientist, with continuous support from divers in the Bermudian community, "a big dent in the population" could be created. In a broader sense, Florida hosts their own Lionfish Derby which reaches a larger audience. Therefore, this will have a better impact as collectively, diving groups will make a larger difference. As stated in the introduction, lionfish rapidly reproduce so as more groups are created and the issue of lionfish becomes more prominent among the public, more diving groups will kill lionfish and slow the growth, but it will still grow due to the high reproductivity rate. If the population were to not exceed 500, the explosion of lionfish would be prevented.

Conclusion

In this investigation, I intended to achieve a highly accurate and informative analysis of the population growth of lionfish in the Atlantic Ocean through the implementation of the numerical method onto a logistic equation. I believe, through consolidation of my results, that my results were not as accurate as expected. This is because of accumulation of errors and also divergence from the sample data. The inaccuracy worsened by the fact that the slope field is increasing very fast, hence producing bad approximations at these points. Approximations also worsen when the slope field is concave down.

Within the exploration, a differential equation for a logistic function was formed after the parameters r and K were determined. These parameters were established from sample data derived from symmetric difference approximation. Furthermore, the logistic model was explored using numerical methods such as Slope Fields and Euler's Method. In calculations using both numerical methods, a general logistic equation was created and then refined to a specific logistic equation. As a result, a particular solution was produced, meaning I created a function that I could plot onto and compare to sample data. Next, approximation calculations involving various step sizes allowed for an analysis of accuracy by producing percentage errors. The point of inflection was then calculated through equating the second derivative to zero, indicating the fastest rate of growth. This result was emphasized by headlines in the news referring to the incident as "The Worst Marine Invasion Ever" in 2013 (Wilcox), providing vital information to the population growth of lionfish.

Personally, the research of lionfish has allowed me to be more mindful of the ecosystem and how fragile it can be when a new species is introduced. I feel I have a richer understanding of numerical analysis within calculus and how the initial point can be used to make predictions. Furthermore, the writing of a mathematical work has allowed me to be descriptive with my explanations, critical and detailed with my analytical processes.

However, this investigation can be further explored using modified Euler's method. From my research, modified Euler's method would provide more accurate approximation for the population growth of lionfish at any time interval. This would be especially beneficial as it builds upon the Euler's method already written in this paper. Some limitations of the work include the dependency of a spreadsheet for some aspects. Perhaps if I could calculate the errors involved and minimize these errors, I would achieve much more accurate approximations.

Based upon the research, the growth of the population of lionfish should be closest to 0 (as per the carrying capacity), however the quantity of lionfish in the Atlantic Ocean is still high. As a result, continuing to encourage the killing and eating of lionfish and supporting lionfish organizations may be necessary. In addition, as I mentioned in the last section, solving for overpopulation's problems is not as simple as fishing a few dozen fish: a more radical solution might be needed. Finally, since many populations grow along an S shaped curve, it is quite clear that there exists a need to intervene earlier if one wants to avoid and prevent a much larger problem in the future. However, it will be significant to consider whether the ecosystem has already adapted to the lionfish, therefore encouraging them to be killed could create a volatile swing in the ecosystem's balance (Bernews). Important knowledge can be gained about invasive species that might be expected to follow a similar population growth to lionfish in the Atlantic. It is worth noting that eventually any invasive species will adjust to the environment and become a part of the ecosystem – but not without major effects on the food chains and reef health.

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EE/RPPF

For use from May/November 2018

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Candidate personal code:

Extended essay - Reflections on planning and progress form

Candidate: *This form is to be completed by the candidate during the course and completion of their EE. This document records reflections on your planning and progress, and the nature of your discussions with your supervisor. You must undertake three formal reflection sessions with your supervisor: The first formal reflection session should focus on your initial ideas and how you plan to undertake your research; the interim reflection session is once a significant amount of your research has been completed, and the final session will be in the form of a viva voce once you have completed and handed in your EE. This document acts as a record in supporting the authenticity of your work. The three reflections combined must amount to no more than 500 words.*

The completion of this form is a mandatory requirement of the EE. It must be submitted together with the completed EE for assessment under Criterion E. As per the 'Protocols for completing and submitting the Reflections on planning and progress form' section of the EE guide, a mark of 0 will be awarded by the examiner for criterion E if the RPPF is blank or the comments are written in a language other than that of the accompanying essay.

Supervisor: *You must have three reflection sessions with each candidate, one early on in the process, an interim meeting and then the final viva voce. Other check-in sessions are permitted but do not need to be recorded on this sheet. After each reflection session candidates must record their reflections and as the supervisor you must sign and date this form.*

First reflection session

Candidate comments:

Living in Bermuda, the dangers of lionfish and their population growth have been highlighted in my community. My supervisor suggested I investigate population growth mathematically using calculus. Despite never learning calculus prior to the Extended Essay, I was eager to challenge myself and decided to independently study a new topic. My initial task was to format data in excel which improved my computer skills significantly. The initial writing of the essay consisted of background information and discussion which was surprisingly straightforward. However, as soon as I read more complex calculus, I realized how difficult it is to grasp the concepts of integration and differentiation and then directly translate the information into more comprehensive terms. I recognized that writing the Extended Essay in Mathematics should clearly and methodically explain the process like a teacher to a student. My research question explores analysis of accuracy and relevancy. Through the use of various mathematical approaches, I will be able to compare and display analysis of my work which will hopefully answer my research question.

Date: 27/09/18

Supervisor initials

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Interim reflection

Candidate comments:

I encountered a couple of issues in terms of content comprehension, so my supervisor recommended some helpful videos and resources to look at. These resources, as well as other resources that I found on my own, allowed me to fully learn the basics before I continued with the differential equation needed for the Essay. I completed practice questions to deepen my knowledge of differentiation and integration. Also, I was not expecting the Essay to require so much out-of-the-syllabus knowledge, but I appreciate that since I have understood the basics, the writing of the work will be easier and more detailed. After some consideration, I developed multiple numerical solutions. My supervisor was pleased to see my progress and how I developed my ideas, suggesting I explore more detailed analysis of Euler's method. I enthusiastically accepted the challenge and hope that it provides depth to my essay.

Date: 10/12/18

Supervisor initials

Final reflection - Viva voce

Candidate comments:

This process has allowed me to understand that I need to be proactive and thorough in order to achieve rich analysis in Mathematics. I learned the importance of grasping the knowledge fully before delving into a concept which allowed me to provide more refined explanations and clearer conclusions. The most challenging aspect of this writing process was also the most interesting: Euler's method. It fascinates me that one point from sample data can be used to relatively accurately predict an entire model. However, I only came to appreciate this method after numerous failed attempts.

The Essay has been one of the most challenging academic papers I have ever completed, yet I have noticed that I am more analytical when looking at Mathematics questions in class. I am confident this work will help me with future Mathematics papers later on, as I know the correct terminology and structure required for such a paper.

If I were to advise a student on a Mathematics Extended Essay, I would encourage that they examine social problems around them and try to find a mathematical explanation for these issues.

Date: 04/03/19

Supervisor initials: